1 Fundamental Coefficients

1.1 Elementary Counting Principles

We begin by collecting a few simple rules that, though obvious, lie at the root of all combinatorial counting. In fact, they are so obvious that they do not need a proof.

Rule of Sum. *If $S = \bigcup_{i=1}^{t} S_i$ is a union of disjoint sets S_i, then $|S| = \sum_{i=1}^{t} |S_i|$.*

In applications, the rule of sum usually appears in the following form: we classify the elements of S according to a set of properties e_i $(i = 1, \dots, t)$ that preclude each other, and set $S_i = \{x \in S : x \text{ has } e_i\}$.

The sum rule is the basis for most recurrences. Consider the following example. A set X with n elements is called an *n-set*. Denote by $S = \binom{X}{k}$ the family of all k-subsets of X. Thus $|S| = \binom{n}{k}$, where $\binom{n}{k}$ is the usual binomial coefficient. For the moment $\binom{n}{k}$ is just a symbol, denoting the size of $\binom{X}{k}$. Let $a \in X$. We classify the members of S as to whether they do or do not contain a: $S_1 = \{A \in S : a \in A\}$, $S_2 = \{A \in S : a \notin A\}$. We obtain all sets in S_1 by combining all $(k-1)$-subsets of $X \smallsetminus a$ with a; thus $|S_1| = \binom{n-1}{k-1}$. Similarly, S_2 is the family of all k-subsets of $X \smallsetminus a$: $|S_2| = \binom{n-1}{k}$. The rule of sum yields therefore the *Pascal recurrence* for binomial coefficients

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad (n \geq k \geq 1)$$

with initial value $\binom{n}{0} = 1$.

Note that we obtain this recurrence without having computed the binomial coefficients.

Rule of Product. *If $S = \prod_{i=1}^{t} S_i$ is a product of sets, then $|S| = \prod_{i=1}^{t} |S_i|$.*

S consists of all t-tuples $(a_1, a_2, \dots, a_t)$, $a_i \in S_i$, and the sets S_i are called the *coordinate sets* .

Example. A sequence of 0's and 1's is called a *word* over $\{0,1\}$, and the number of 0's and 1's the *length* of the word. Since any coordinate set S_i has two elements, the product rule states that there are 2^n n-words over $\{0,1\}$. More generally, we obtain r^n words if the alphabet A contains r elements. We then speak of n-words over the *alphabet* A.

Rule of Bijection. *If there is a bijection between S and T, then $|S| = |T|$.*

The typical application goes as follows: Suppose we want to count S. If we succeed in mapping S bijectively onto a set T (whose size t is known), then we can conclude that $|S| = t$.

Example. A simple but extremely useful bijection maps the powerset 2^X of an n-set X, i.e., the family of *all* subsets of X, onto the n-words over $\{0,1\}$. Index $X = \{x_1, x_2, \ldots, x_n\}$ in any way, and map $A \subseteq X$ to $(a_1, a_2, \ldots, a_n)$ where $a_i = 1$ if $x_i \in A$ and $a_i = 0$ if $x_i \notin A$. This is obviously a bijection, and we conclude that $|2^X| = 2^n$. The word $(a_1, \ldots, a_n)$ is called the *incidence vector* or *characteristic vector* of A.

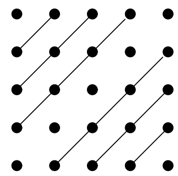
The rule of bijection is the source of many intriguing combinatorial problems. We will see several examples in which we deduce by algebraic or other means that two sets S and T have the same size. Once we know that $|S| = |T|$, there exists, of course, a bijection between these sets. But it may be and often is a challenging problem to find in the aftermath a "natural" bijection based on combinatorial ideas.

Rule of Counting in Two Ways. *When two formulas enumerate the same set, then they must be equal.*

This rule sounds almost frivolous, yet it often reveals very interesting identities. Consider the following formula:

$$\sum_{i=1}^{n} i = \frac{(n+1)n}{2}\,. \tag{1}$$

We may, of course, prove (1) by induction, but here is a purely combinatorial argument. Take an $(n+1) \times (n+1)$ array of dots, e.g., for $n = 4$:



The diagram contains $(n+1)^2$ dots. But there is another way to count the dots, namely by way of diagonals, as indicated in the figure. Clearly, both the upper and lower parts account for $\sum_{i=1}^n i$ dots. Together with the middle diagonal this gives $2\sum_{i=1}^n i + (n+1) = (n+1)^2$, and thus $\sum_{i=1}^n i = \frac{(n+1)n}{2}$.

We even get a bonus out of it: the sum $\sum_{i=1}^n i$ enumerates another quantity, the family S of all *pairs* in the $(n+1)$-set $\{0,1,2,\ldots,n\}$. Indeed, we may partition S into disjoint sets S_i according to the *larger* element i, $i=1,\ldots,n$. Clearly, $|S_i| = i$, and thus by the sum rule $|S| = \sum_{i=1}^n i$. Hence we have the following result: the number of pairs in an n-set is $\binom{n}{2} = \frac{n(n-1)}{2}$.

The typical application of the rule of counting in two ways is to consider incidence systems. An *incidence system* consists of two sets S and T together with a relation I. If aIb, $a \in S$, $b \in T$, then we call a and b *incident*. Let $d(a)$ be the number of elements in T that are incident to $a \in S$, and similarly $d(b)$ for $b \in T$. Then

$$\sum_{a\in S} d(a) = \sum_{b\in T} d(b)\,.$$

The equality becomes obvious when we associate to the system its *incidence matrix* M. Let $S = \{a_1,\ldots,a_m\}$, $T = \{b_1,\ldots,b_n\}$, then $M = (m_{ij})$ is the $(0,1)$-matrix with

$$m_{ij} = \begin{cases} 1 & \text{if } a_i I b_j, \\ 0 & \text{otherwise.} \end{cases}$$

The quantity $d(a_i)$ is then the i-th row sum $\sum_{j=1}^n m_{ij}$, $d(b_j)$ is the j-th column sum $\sum_{i=1}^m m_{ij}$. Thus we count the total number of 1's once by row sums and the other time columnwise.

Example. Consider the numbers 1 to 8, and set $m_{ij} = 1$ if i divides j, denoted $i \mid j$, and 0 otherwise. The incidence matrix of this divisor relation looks as follows, where we have omitted the 0's:

	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2		1		1		1		1
3			1			1		
4				1				1
5					1			
6						1		
7							1	
8								1

The j-th column sum is the number of divisors of j, which we denote by $t(j)$ thus, e.g., $t(6) = 4$, $t(7) = 2$. Let us ask how many divisors a number from 1 to 8 has on *average*. Hence we want to compute $\overline{t}(8) = \frac{1}{8}\sum_{j=1}^{8} t(j)$. In our example $\overline{t}(8) = \frac{5}{2}$, and we deduce from the matrix that

n	1	2	3	4	5	6	7	8
$\overline{t}(n)$	1	$\frac{3}{2}$	$\frac{5}{3}$	2	2	$\frac{7}{3}$	$\frac{16}{7}$	$\frac{5}{2}$

How large is $\overline{t}(n)$ for arbitrary n? At first sight this appears hopeless. For prime numbers p we have $t(p) = 2$, whereas for powers of 2, say, an arbitrarily large value $t(2^k) = k + 1$ results. So we might expect that the function $\overline{t}(n)$ shows an equally erratic behavior. The following beautiful application of counting in two ways demonstrates that quite the opposite is true!

Counting by columns we get $\sum_{j=1}^{n} t(j)$. How many 1's are in row i? They correspond to the multiples of i, $1 \cdot i, 2 \cdot i, \ldots$, and the last multiple is $\lfloor \frac{n}{i} \rfloor i$. Our rule thus yields

$$\overline{t}(n) = \frac{1}{n}\sum_{j=1}^{n} t(j) = \frac{1}{n}\sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor \sim \frac{1}{n}\sum_{i=1}^{n} \frac{n}{i} = \sum_{i=1}^{n} \frac{1}{i},$$

where the error going from the second to the third sum is less than 1. The last sum $H_n = \sum_{i=1}^{n} \frac{1}{n}$ is called the n-th *harmonic number*. We know from analysis (by approximating $\log x = \int_1^x \frac{1}{t}\,dt$) that $H_n \sim \log n$, and obtain the unexpected result that the divisor function, though locally erratic, behaves on average extremely regularly: $\overline{t}(n) \sim \log n$.

You will be asked in the exercises and in later chapters to provide combinatorial proofs of identities or recurrences. Usually, this means a combination of the elementary methods we have discussed in this section.

Exercises

1.1 We are given t disjoint sets S_i with $|S_i| = a_i$. Show that the number of subsets of $S_1 \cup \ldots \cup S_t$ that contain at most one element from each S_i is $(a_1+1)(a_2+1)\cdots(a_t+1)$. Apply this to the following number-theoretic problem. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ be the prime decomposition of n then $t(n) = \prod_{i=1}^{t}(a_i+1)$. Conclude that n is a perfect square precisely when $t(n)$ is odd.

▷ **1.2** In the parliament of some country there are 151 seats filled by 3 parties. How many possible distributions (i, j, k) are there that give no party an absolute majority?

1.3 Use the sum rule to prove $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$, and to evaluate $\sum_{k=1}^{n}(n-k)2^{k-1}$.

1.4 Suppose the chairman of the math department stipulates that every student must enroll in exactly 4 of 7 offered courses. The teachers give the number in their classes as $51, 30, 30, 20, 25, 12$, and 18, respectively. What conclusion can be drawn?

▷ **1.5** Show by counting in two ways that $\sum_{i=1}^{n} i(n-i) = \sum_{i=1}^{n} \binom{i}{2} = \binom{n+1}{3}$.

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1.6 Join any two corners of a convex n-gon by a chord, and let $f(n)$ be the number of pairs of crossing chords, e.g., $f(4) = 1$, $f(5) = 5$. Determine $f(n)$ by Pascal's recurrence. The result is very simple. Can you establish the formula by a direct argument?

1.7 In how many ways can one list the numbers $1, 2, \ldots, n$ such that apart from the leading element the number k can be placed only if either $k-1$ or $k+1$ already appears? Example: 324516, 435216, but not 351246.

▷ **1.8** Let $f(n, k)$ be the number of k-subsets of $\{1, 2, \ldots, n\}$ that do not contain a pair of consecutive integers. Show that $f(n, k) = \binom{n-k+1}{k}$, and further that $\sum_{k=0}^{n} f(n, k) = F_{n+2}$ (Fibonacci number).

1.9 Euler's φ-function is $\varphi(n) = \#\{k : 1 \le k \le n,\ k \text{ relatively prime to } n\}$. Use the sum rule to prove $\sum_{d|n} \varphi(d) = n$.

1.10 Evaluate $\sum_{i=1}^{n} i^2$ and $\sum_{i=1}^{n} i^3$ by counting configurations of dots as in the proof of $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.